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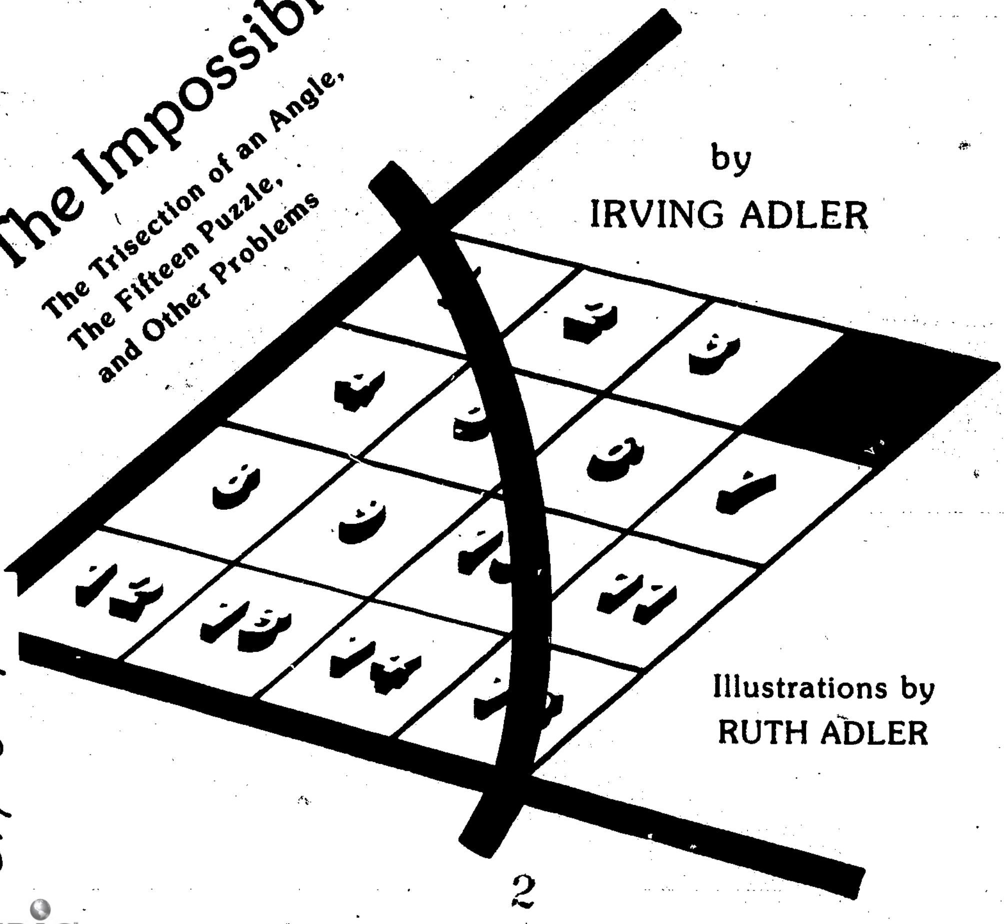
ABSTRACT

The material in this reprint, with minor editorial changes, is from the chapter "Doing the Impossible" in MONKEY BUSINESS by Irving Adler. This 25-page booklet contains brief accounts of historical attempts to prove impossible problems in mathematics. The mathematical recreations in this booklet of geometric constructions include the trisection problem, the "fifteen" puzzle, the "64" puzzle, doubling the cube, squaring the circle, and perpetual motion machines. The latter problem extends from geometric construction to involve rules of nature. (JBW)

The Impossible in Mathematics

The Trisection of an Angle, The Fifteen Puzzle, and Other Problems

by
IRVING ADLER



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National Council of Teachers of Mathematics

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SUMMARY: Brief accounts of historical attempts to prove impossible problems in mathematics, such as the trisection problem, the "fifteen" and "64" puzzles, squaring the circle, etc.

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The Trisection Problem

THE YEAR 1931 was a year of turmoil and great change. The United States was in the grip of a depression, with many factories shut down, and millions of people unemployed. In Germany, Hitler's brown-shirted Nazis were preparing to seize power. In the Far East, Japan had begun her invasion of China. But, one day in August, the newspapers pushed all these world-shaking events to one side, to make room for an exciting headline: *Trisection of Angle by Euclid accomplished after 2500 Years.* Of course they didn't mean that Euclid had done it. Euclid had been dead all these twenty-five hundred years, although his geometry textbook lived on. The newspapers meant that at last the angle had been trisected by the *methods* of Euclid. The man who did it, the papers told us, was the Very Reverend Jeremiah Joseph Callahan, President of Duquesne University. Reporters rushed to interview him, to find out how he had cracked the problem that had baffled mathematicians for so many centuries. The United Press scooped its competitors by running a first-hand account written and signed by Father Callahan himself.

But a shadow fell across Father Callahan's accomplishment. The newspapers reported that other mathematicians were skeptical. Were they merely being cautious, waiting to see the details of Callahan's method before they would pass judgment on it? No, they were not being cautious. In fact, many expressed their opinions freely. Although Callahan had not yet published his construction, they said that they were sure it would prove to be wrong. Were they being pig-headed, then, unwilling to admit that somebody had succeeded where they themselves had failed? No, they were not pig-headed. The reporters found no signs of indignation, hurt pride or professional jealousy. The mathematicians patiently explained to the reporters that the problem of trisecting an angle was not an unsolved problem waiting for somebody with daring and imagination to tackle it. *This problem had been solved in 1837.* But the solution was a proof that it is *impossible* to trisect an angle by the methods of Euclid. That is why they were sure that Father Callahan was wrong.

The shadow across Callahan's fame grew and swallowed it up when he finally published his construction in December 1931. The mathematicians were right. Callahan had not trisected the angle. To trisect an angle means to start with any angle and then divide it into three equal parts to produce angles one-third the size of the original angle. What Callahan had done was the direct opposite. He had started with an angle and then tripled it, producing an angle three times the size of the original angle. Tripling an angle is an easy construction that every high school geometry student knows how to do. Callahan's roundabout way of doing it wasn't even an improvement over the usual method of solving this simple problem.

Father Callahan is not the only person who has claimed that he trisected the angle in recent years. Every few years somebody else comes out with a new construction. And every time it turns out to be wrong. Then a new crop of hopeful amateurs starts trying where the others have failed. Some of them make the attempt without knowing that the construction has been proved impossible. But most of them try *because* they have heard it is impossible. When they are told it is impossible, they take it as a challenge. It is as if someone said to them, "I dare you to do it." And what person with courage will not take a dare, especially if he can do it sitting safely at his desk, using no deadlier weapons than a pencil and paper and a ruler and compasses?

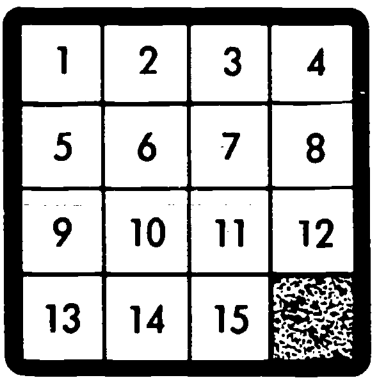
The reason why so many people keep trying to do the impossible in geometry is because they do not believe anything is really impossible. They have unlimited faith in what human ingenuity can accomplish. They see that many things that were impossible for our ancestors are now everyday occurrences for us. We fly through the air in giant planes that can circle the globe. We erect buildings as tall as mountains. By radio or telephone, we speak to people thousands of miles away. Through phonograph records and moving picture sound tracks we can hear the voices of people who are dead. All these miracles of human invention have convinced them that there is no obstacle that science and industry cannot overcome. We can solve any problem, they think, if we just work at it hard enough and long enough. This is the spirit in which the Navy's construction battalions, the Seabees, adopted their famous slogan, "The difficult we do immediately. The impossible takes a little longer."

The spirit of not backing down before difficulties is a good thing. Without it mankind would never have climbed from savagery to civilization. But when it is applied to the problem of trisecting an angle, it can lead only to wasted effort. There are some things that are *really impossible*, and trisecting an angle with a straight edge and compasses is one of them.

To see why some things are really impossible, let us begin with a very simple example. Suppose a newspaper, reporting a baseball game, gave the final score as 8½ to 3. Everybody would know immediately that the report was wrong, because it is impossible to have such a score in baseball. It is impossible because of the rules by which baseball scores are tallied. *Under the rules of the game, the only scores that are possible are whole numbers.* So, when the game is played according to the rules, no team could ever get a score of 8½.

The Fifteen Puzzle

We can find another helpful example of the impossible in the famous *Fifteen Puzzle* which has fascinated young and old for almost a hundred years. The puzzle consists of a square frame that has room for sixteen square blocks, arranged in four rows with four blocks in each row. But there are only fifteen blocks in the frame, so that there is an empty space into which a block can fit. The blocks are numbered from one to fifteen. To play the game, you start with the blocks arranged in numerical order, with the numbers 1 to 4 in the first row, 5 to 8 in the second row, 9 to 12 in the third row, 13 to 15 in the last row, and the blank space in the lower right-hand corner. The object of the game is to move the blocks around to get some new arrangement of the numbers decided on in advance. But the moving must be done *according to certain rules:* You may not lift the blocks out of the frame. You may move the blocks only one at a time by sliding into the empty space a block that lies right next to it. And you must end up with the blank space in the lower right-hand corner again.



STARTING POSITION

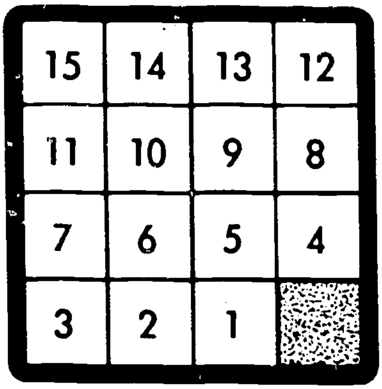
YOU CAN GET THIS....

15	14	13	12
11	10	9	8
7	6	5	4
3	1	2	

When you try the puzzle, you soon discover that there are some arrangements that you can reach without any trouble at all. But there are other arrangements that you cannot get no matter how hard you try. You get into trouble, for example, if you try to arrange the numbers in reverse order, with the 15 in the upper left-hand corner, and the blank space in the lower right-hand corner. You may succeed in getting the numbers from 15 to 3 in the right positions, but the last two numbers, the 1 and the 2, stubbornly insist on taking each other's places. When you fail to arrange the numbers in reverse order, it doesn't mean that you aren't smart enough or didn't try hard enough to succeed. You fail because it is actually impossible to get that arrangement. Just as the rules of baseball make whole number scores possible and fractional scores impossible, the rules of this puzzle make some arrangements possible and other arrangements impossible to reach.

The fact that there are impossible arrangements for the Fifteen Puzzle is not as obvious as the fact that fractional scores are impossible in baseball. But it is easy to prove by a little careful thinking. The proof is given here as an example of the method by which a mathematician can prove that something is really impossible.

BUT NOT THIS....



To understand the proof we first have to get acquainted with some simple facts about numbers. Let us notice, to begin with, that the whole numbers are divided into two families, the *even numbers* and the *odd numbers.* We get the even numbers by starting with 0 and counting by two's. This gives us the series of numbers, 0, 2, 4, 6, 8, 10, 12, and so on. We get the odd numbers by adding 1 to each of the even numbers. This gives us the series of numbers, 1, 3, 5, 7, 9, 11, 13, and so on. Now we observe these additional facts:

Rule 1: When you add an odd number to an even number, the result is an odd number. For example, when you add 3 to 8, the result is 11, an odd number.

Rule 2: When you subtract an odd number from an even number, the result is an odd number. For example, when you subtract 3 from 8, the result is 5, an odd number.

Rule 3: When you add an odd number to an odd number, the result is an even number. For example, when you add 3 to 9, the result is 12, an even number.

Rule 4: When you subtract an odd number from an odd number, the result is an even number. For example, when you subtract 3 from 9, the result is 6, an even number.

Next we notice a simple but important fact about arrangements of numbers. Let us arrange a series of numbers in a line, from left to right. We shall say a number is *before* another number, or *precedes* it, if it is to the left of that number. We shall say a number is *after* another number, or *follows* it, if it is to the right of that number. For example, in the arrangement 1, 3, 2, 5, 4, the 3 is before the 2, and the 3 and 2 both follow the 1.

Now we compare each of the numbers with those that come after it. Whenever a number is followed by a smaller number, we say that the pair of numbers forms an *inversion.* In the arrangement 1, 3, 2, 5, 4, the 1 is not followed by any smaller numbers, so it is not part of any inversion. The 3 is followed by the smaller number 2, so the pair 3, 2 is an inversion. The pair 5, 4 is another inversion. This arrangement has two inversions altogether. The arrangement 1, 2, 3, 4, 5 has no inversions, because no number is followed by a smaller number. The arrangement 5, 4, 3, 2, 1 has many inversions which we can count up in this way: The 5 is followed by four smaller numbers, giving us four inversions. The 4 is followed by three smaller numbers, giving us three more inversions. The 3 is followed by two smaller numbers, giving us two more inversions. The 2 is followed by one smaller number, giving us one more inversion. The total number of inversions is four plus three plus two plus one, which adds up to ten inversions.

In any arrangement of numbers, we find the number of inversions by counting how many smaller numbers follow the first number in the line, how many smaller numbers follow the second number, and so on down the line, until we have the total count. When the numbers from 1 to 15 are arranged in order of size, there are no inversions, so we say the number of inversions is 0. When the numbers are arranged in reverse order, the number of inversions is 14 plus 13 plus 12 plus 11 plus 10 plus 9 plus 8 plus 7 plus 6 plus 5 plus 4 plus 3 plus 2 plus 1. Add these numbers up, and we find that the reverse arrangement has 105 inversions. Notice that 105 is an *odd* number. This fact is important in the proof that it is impossible to get this arrangement when we move the numbers in the Fifteen Puzzle according to the rules.

Now let us examine a simple arrangement of four numbers, like 8, 3, 2, 10, and see what happens to the number of inversions if the first number, the 8, is picked up and placed after the others, so that it becomes the last number. In the original arrangement, the 8 was before the 3 and 2, forming two inversions. In the new arrangement, the 8 follows the 3 and 2, so these two inversions have been removed. In the original arrangement, the 8 was before the 10, and this is not an inversion. In the new arrangement, the 8 follows the 10, so a new inversion has been introduced. When we put the 8 last, the arrangement loses two inversions and gains one inversion. The total change in the number of inversions in this case is the loss of one inversion. Notice that inversions like 3, 2, which do not involve the number 8, are not changed, because the 3 and 2 did not change places.

Now let us do the same thing with any arrangement of any four numbers in a line. In the original arrangement, the first number is followed by three others. If all three are smaller than the first number, they form three inversions with the first number. If only two of them are smaller, and the third is larger, the two smaller ones form inversions with the first number, while the larger one does not. If only one of them is smaller, and the other two are larger, the smaller one forms an inversion with the first number, but the two larger ones do not. If none of the three numbers are smaller, they form no inversions with the first number at all. Now, when we place the first number last, it follows the numbers that it used to precede. This reverses the order in the pair that it forms with each of these numbers. As a result, wherever it formed an inversion in the original arrangement, this inversion is lost. Wherever it formed no inversion in the original arrangement, an inversion is gained. All the possibilities are shown in the table below:

ORIGINAL ARRANGEMENT		NEW ARRANGEMENT	
Inversions	3	Inversions lost	3
Not an inversion	0	Inversions gained	0
		Total change:	3 lost
Inversions	2	Inversions lost	2
Not an inversion	1	Inversions gained	1
		Total change:	1 lost
Inversions	1	Inversions lost	1
Not an inversion	2	Inversions gained	2
		Total change:	1 gained
Inversions	0	Inversions lost	0
Not an inversion	3	Inversions gained	3
		Total change:	3 gained

So whenever the first of four numbers is placed last, the change in the number of inversions is a gain of 1 or 3, or a loss of 1 or 3. But 1 and 3 are odd numbers. So we see that in each of the four possible cases, an *odd number* is added to or subtracted from the number of inversions.

Now we are ready to analyze the Fifteen Puzzle. If we were permitted to move the blocks in any way we please, we would be able to get any arrangement of the fifteen numbers in the frame. All we would have to do is lift the blocks out of the frame, and then put them back in the arrangement we want. *But we are not free to move the blocks in any way we please. We must follow the rules of the game.* These rules, as we shall see, make some arrangements impossible to reach.

One rule is that we move the blocks one at a time, by sliding a block into the empty space. A move may be of four different types. We may slide a block to the left into the empty space, or we may slide a block to the right into the empty space. We may also slide a block down into the space from the line above it, or we may slide a block up into the space from the line below it. Let us examine the effect that each move has on the arrangement of the numbers in the frame. Although the numbers are arranged in four lines, it is like an arrangement in one line, because we can think of the second line as being after the first, the third line as being after the second, and the fourth line as being after the third. Then one number is after another number if it follows it in the same line, or if it is in one of the later lines. Since we have a way of judging whether one number is before or after another, we can count the number of inversions in any arrangement. Now let us see what happens to the number of inversions whenever we make a move according to the rules. If we move a block to the right or the left, the order of the numbers in the frame is not changed. So *a move to the left or right has no effect on the number of inversions.* To see the effect of a move up or down, look at the diagram below.

1	2	3	4
5	10	6	8
9		7	12
13	14	11	15

Suppose we move the 10 into the empty space below it. Before the move, the 10 is the first of the four numbers, 10, 6, 8, 9. After the move, it becomes the last of these four numbers. We know already that as a result of this type of change, the number of inversions is changed by an odd number. In this case it is decreased by three. *Each move up or down either adds an odd number of inversions, or subtracts an odd number from the number of inversions.*

Now let's figure out the total effect of *all the moves* we make when we play the game. We begin with the numbers arranged in order of size, from 1 to 15. In this arrangement, the number of inversions is 0, which is an even number. We can disregard the moves to the right or left, because they do not change the number of inversions. The first up or down move adds an odd number to the number of inversions. All up or down moves after the first may add or subtract an odd number. We can keep track of the results by using the four rules about odd and even numbers that we discovered on page 6. The first up or down move starts with an even number of inversions, and adds an odd number, so the result is an odd number of inversions. The second move up or down starts with this odd number of inversions, and adds or subtracts an odd number. The result is an even number. The third move changes it back to an odd number. The number of inversions keeps changing back and forth from even to odd, and from odd to even. Notice that after two moves, four moves, six moves, and so on, the number of inversions is even. So we have this important rule about the puzzle: *The arrangement of the numbers after an even number of up or down moves has an even number of inversions.*

Another of the rules of the game is that the blank space must begin in the lower right-hand corner, and it must end in the lower right-hand corner. This rule is important, because it places a restriction on the number of up and down moves we may make. Whenever a block moves down, the empty space moves up to take its place. Whenever a block moves up, the empty space moves down to take its place. So we can count the number of up or down moves we make by counting how many times the empty space moves up or down. After all our moves, the empty space ends up where it started. This means that for every time that it moves up, it must move down again in order to get back. So the up and down moves come in pairs, with each up move balanced by a down move. Then we can count them by two's. But when we count by two's, the result is always an even number. So we see that the rule that the empty space must end where it begins forces us to make an even number of up and down moves. But after an even number of up and down moves, the number of inversions is even. So the only arrangement of the numbers that we can reach when we follow the rules of the game is one that has an even number of inversions. This means that *it is impossible to reach arrangements that have an odd number of inversions.* We have seen that the reverse arrangement, from 15 to 1, has 105 inversions, which is an odd number. So this arrangement is impossible to get.

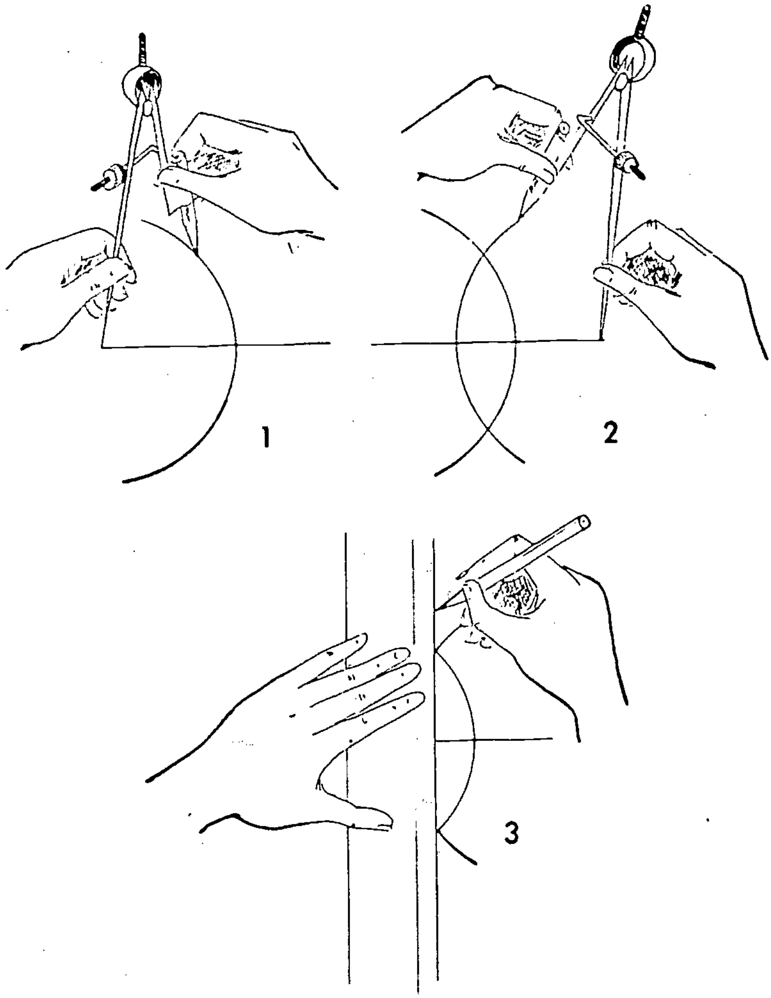
The situation can be reversed if we change the beginning arrangement. If we start with an arrangement that has an odd number of inversions, then we can reach all other arrangements that have an odd number of inversions. But then the positions that have an even number of inversions become impossible. A recent model of the Fifteen Puzzle is made out of plastic squares that slide on tracks inside a square frame. The tracks serve to enforce the rules of the game, because they prevent any movement of the squares except sliding. But many young people get around this restriction by forcing one of the squares off the track, and then forcing it on again in another position. In this way they make the impossible arrangements possible. But, at the same time, they make the possible arrangements impossible, if they observe the rules of the game once more.

The "64" Puzzle

We can use our knowledge about odd and even numbers to analyze another problem now, the famous "64" puzzle which is a favorite with many teachers. If a teacher has some clerical work to do, and wants to keep her class busy and out of mischief while she does it, she may offer a bonus of ten points to any pupil who finds five odd numbers that add up to 64. The class usually accepts the challenge and gets to work on the problem at once. The teacher is sure that they will not solve the problem too soon. She is even sure that she will not have to give anybody the ten-point bonus. To see why, let us find out what happens when we add a series of odd numbers. When we add the first two odd numbers, the result is even. When we add the third odd number to this sum, the result is odd. When we add the fourth odd number the result is even. When we add the fifth odd number the result is odd. So, when we add any five odd numbers, the result *must be odd.* It is impossible for the result to be even. But 64 is an even number, *so it is impossible for five odd numbers to add up to 64.* Even if the class works on the problem until every boy in the class has a beard four feet long, and every girl in the class is a grandmother, they will never be able to solve it.

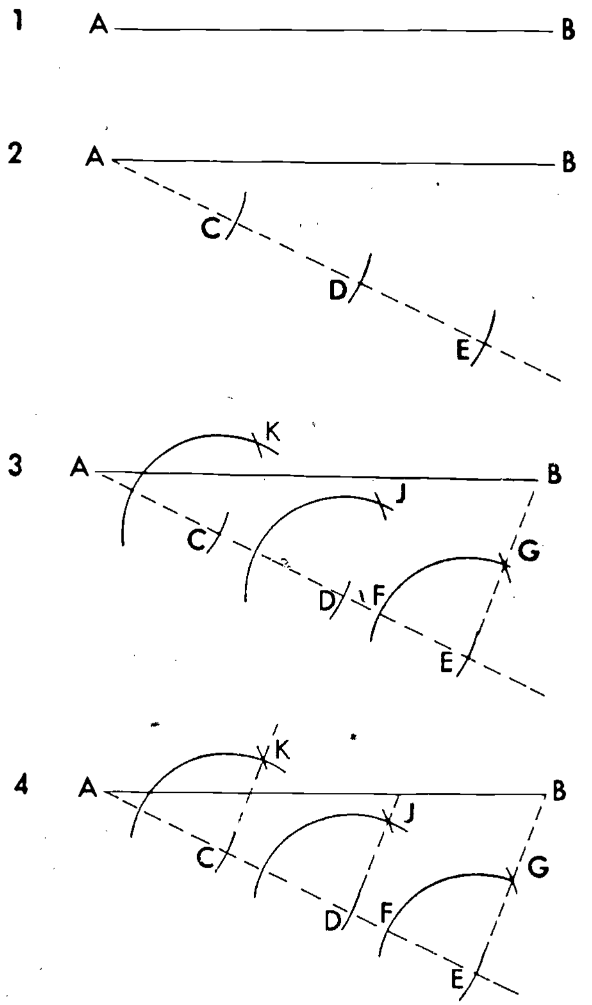
The three examples we have examined help us understand why there are some things that are really impossible. Whenever we are required to do things according to definite rules, the rules put a restriction on the kind of results we can get. *Results which are not permitted by the rules are impossible to get, as long as we follow the rules.* In baseball, the rules of scoring make a fractional score impossible. In the Fifteen Puzzle, the rules about how to begin and end, and how to move the squares, make an arrangement with an odd number of inversions impossible. In the "64" puzzle, the rule that we use an odd number of odd numbers makes it impossible to get an even number as the sum. By studying the consequences of the rules of a game, it is possible *to prove* that certain results are impossible. This is what has happened with the problem of trisecting an angle.

Over two thousand years ago, Greek mathematicians found that they could do many constructions by using only two simple instruments: a straight edge, for drawing straight lines between points, and compasses for drawing circles. They could use these instruments, for example, to bisect a line, or divide it into two equal parts. The method is shown in the series of drawings on page 14. The two circles that



HOW TO BISECT A LINE

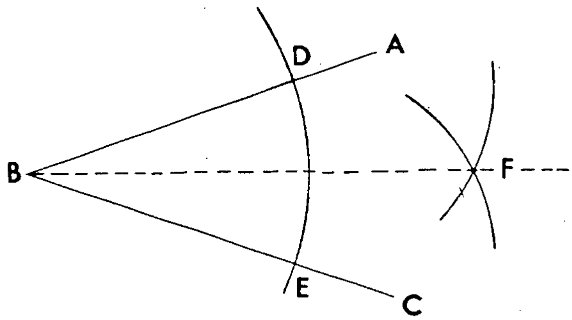
are shown in this drawing both have the same radius. They also knew how to trisect a line, or divide it into three equal parts. The method used is shown in the drawings on page 15. AB is the line to be trisected. Another line is drawn



HOW TO TRISECT A LINE

from A, and the compasses are used to cut off three equal pieces on it. The last point of division, E, is joined to B, forming an angle at E. With E as center, an arc of a circle is drawn, cutting the sides of the angle at F and G. The same radius is used to make arcs that have C and D as centers. Then using FG as radius, equal pieces are cut off on all three arcs. This locates the points J and K. When the lines DJ and CK are drawn, they trisect the line AB.

The Greek mathematicians also found it easy to bisect an angle. The method is shown in the drawing below. To bisect angle ABC, first an arc is drawn with B as center. This arc cuts the sides of the angle at D and E. Then with D and E as centers two arcs are drawn with equal radii. These arcs cross each other at F. When the line BF is drawn, it bisects



HOW TO BISECT AN ANGLE

the angle. Since they found it so easy to bisect the angle, they naturally tried to trisect it, too. But then they ran into trouble. They never succeeded as long as they used only a straight edge or compasses. Another way of saying this is that they never succeeded as long as they drew only straight lines and circles. They did succeed in trisecting the angle by using other curves that are more complicated than the circle. But they felt as though they were cheating when they did. They thought it should be possible to manage with straight lines and circles alone, so they kept trying. This is how the problem of trisecting an angle arose. Notice that the problem is not merely to divide an angle into three equal parts. The Greeks solved that problem a long time ago with their special curves. The problem is to do it in a particular way, *using only a straight edge and compasses.* This makes the problem like a game that must be played according to definite rules.

For thousands of years mathematicians tried to trisect an angle according to the rules. They did not succeed. Their failure led some to suspect that the construction might be impossible. To check their suspicions they began to investigate the meaning of the rule that only a straight edge and compasses may be used. To explore its meaning, they made use of a discovery by the great French philosopher and mathematician Descartes. Descartes had found that every geometry problem can be turned into an algebra problem. Then, instead of working with lines and curves, you work with numbers and equations. This led the German mathematician, Gauss, and the American mathematician Wantzel to ask what kind of equation belongs to a construction in which you use only straight lines and circles. The answer they found to this question makes it possible to identify which constructions are possible and which are impossible.

They found first that the equation must belong to a family of equations known as algebraic equations, in which powers of the unknown are multiplied by whole numbers and added or subtracted to give zero. A typical equation of this family looks like this: $4x^5 - 3x^2 + 2x + 7 = 0$. The highest power of the unknown that appears in the equation is called the degree of the equation. The degree of the equation shown above is 5. They found, secondly, that if a construction can be carried out by means of a straight edge and compasses, the degree of its equation has to be a power of 2. The powers of 2 are the numbers we get when we multiply 1 repeatedly by 2. These numbers are 1, 2, 4, 8, 16, and so on. The degree of the equation that belongs to the problem of trisecting an angle happens to be 3. Since 3 is not a power of 2, *the construction is impossible.* The proof of this result was first published by Wantzel in 1837.

Doubling the Cube

Wantzel's proof also disposed of another famous problem that has come down to us from ancient times; that of doubling a cube. This problem has a curious history, because it had its origin in an epidemic of typhoid fever. The fever struck the city of Athens in the year 430 B.C. Thousands of people were sick. Many died, and more were dying. The terrified inhabitants turned to their gods for help. They sent their officials to Delos to get the advice of the oracle at the temple of Apollo. The oracle was a priest who was supposed to have special powers of seeing into the future. The oracle told the Athenians that Apollo would help them if they would double the altar that stood in the temple. The altar was a block of stone cut in the shape of a cube. The officials measured the altar and ordered a new one made with edges twice as long as the edges of the old one. After the new altar was installed, the epidemic grew worse. The officials hurried back to Delos to find out what they should do now. Again the oracle advised them that the plague would leave them if they would double the old altar of Apollo. They thought they had already done so, but evidently Apollo was displeased, so they decided that they must have done something wrong. Now they turned to their mathematicians for advice. What did the oracle mean when he told them to double the altar? The mathematicians decided that Apollo wanted them to double the *volume* of the altar, not the length of its edges. But to make an altar with double the volume, and still shaped like a cube, it was necessary to find out first how long its edge should be. So, to save themselves from the fever, the unfortunate people of Athens had to solve this mathematical problem: Starting with any cube, construct the edge of a cube that has double the volume of the original cube. Naturally, the Athenian mathematicians tried to solve it with their favorite instruments, the straight edge and compasses. They failed. Fortunately Apollo relented, and the epidemic came to an end anyhow. It is a good thing that the ending of the epidemic did *not* depend on their solving the problem. For over two thousand years other mathematicians also tried to solve the problem. None of them succeeded.

In 1837, Wantzel's proof showed why. The equation that belongs to the problem of doubling the volume of a cube is $x^3 - 2 = 0$. The degree of this equation is 3. Since 3 is not a power of 2, it is impossible to solve the problem by means of a straight edge and compasses.

Squaring the Circle

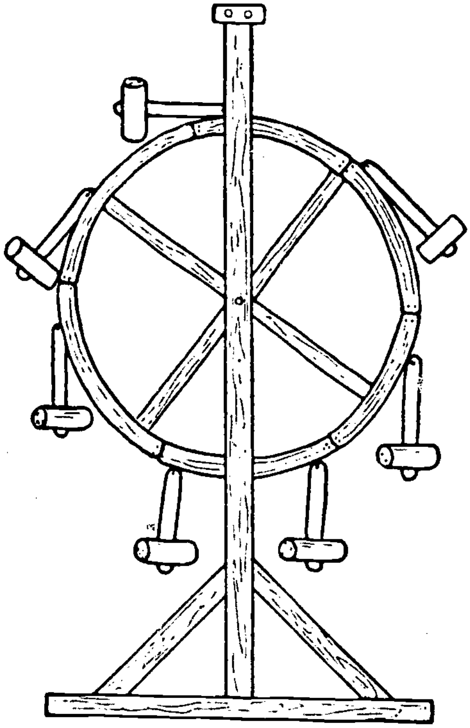
There is another famous geometric construction that defied solution for thousands of years. This problem, usually referred to as "squaring the circle," grew out of early attempts to measure the area of a circle. In the year 1650 B.C., an Egyptian mathematician gave these directions for doing it: To transform a circle into a square of equal area, cut off one-ninth of the diameter of the circle. "The square on the remainder," he said, "will equal the area of the circle." Later mathematicians were not satisfied with these directions. The square that you get by this method is almost equal to the circle, but not quite. So they tried to find a method that would be exact. Stated in the language of geometry, their problem was to start with any circle, and, using a straight edge and compasses, to construct the side of a square that has the same area. The work of Gauss and Wantzel showed that this is possible only if the number π, which is needed to calculate the area of a circle, can satisfy an algebraic equation whose degree is a power of 2. Their work paved the way for the final solution of the problem. As in the case of trisecting an angle and doubling a cube, the problem was solved by proving that the construction with straight edge and compasses is impossible. This was done in 1882, when Lindemann showed that the number π cannot satisfy any algebraic equation at all, so it certainly could not satisfy one whose degree is a power of 2.

Perpetual Motion Machines

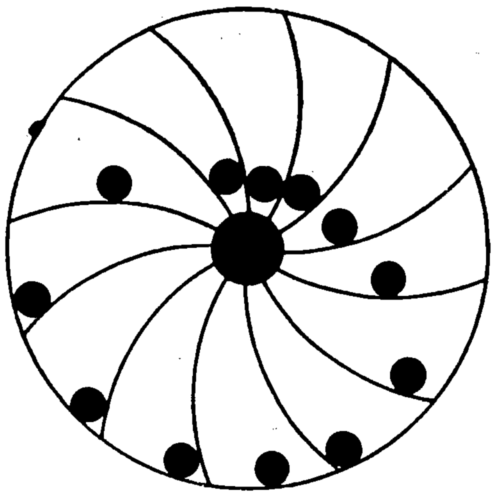
Trisecting an angle, doubling a cube, and squaring a circle turned out to be impossible to do because of the rule which restricted the way in which people tried to do them. But, since men had made up the rule, they were also free to change the rule. Once they permitted the use of other instruments besides a straight-edge and compasses, the constructions become possible, and have been carried out by many different methods. But there is another famous construction problem that cannot be tackled in the same way. This construction, too, is impossible because of the rules that govern attempts to do it. But in this case we cannot change the rules. They are not man-made rules. They are rules of nature. The problem is that of building a *perpetual motion machine*.

We build machines to help us in our work. But machines are hungry servants. We have to feed them in order to keep them working. Water wheels must be fed flowing water. Steam engines have to be fed coal. Electric motors have to be fed an electric current. Feeding the machines takes effort, so the machines don't free us entirely from the burden of work. This weakness of machines was very distrubing to early inventors. They kept improving the machines so that they would deliver more work with less human effort. Then some inventors got the idea of making a perfect machine that would require *no human effort at all*. They tried to design a machine which would need only to be started. Then, after that it would keep moving all by itself, forever.

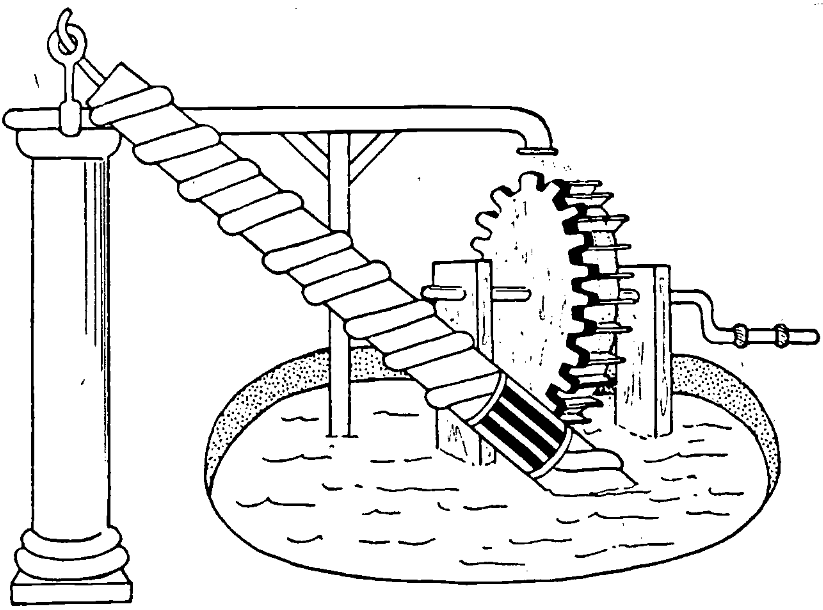
The earliest design we know of for a perpetual motion machine was published in the thirteenth century by Vilard de Honnecourt. The drawing below shows what his machine was like. A wooden frame supported a large wheel with four spokes. Seven hammers were hinged to the rim of the wheel so they could swing freely. When the wheel was given a turn, the hammer at the top was carried around with it and fell. The force of the fall was supposed to give the wheel another push, so that the next hammer would be turned far enough to fall. In this way, each hammer, as it fell, would help to turn the wheel. Since another hammer always came to the top to take the place of the one that just fell, this was supposed to go on forever.



A design using the same principle is found among the sketches left by Leonardo da Vinci, the famous artist and inventor who lived from 1452 to 1519. Leonardo also used falling weights, but he put them inside the wheel instead of attaching them to the rim. In his design, the spokes of the wheel are curved blades that serve as tracks for a rolling ball. The ball starts near the hub of the wheel, and, as it falls, it rolls out toward the rim. Meanwhile the weight of the falling ball turns the wheel. But as the wheel turns, the blade at the bottom scoops its ball up again. The ball gradually rolls back towards the hub, and then is ready to fall all over again. In this machine the inside of the wheel would have to be enclosed between two flat sheets to keep the balls from falling out.



As the years went by, perpetual motion machines became more and more complicated. They were made with gears and springs. Some used magnets. Some were chemically operated. Others replaced falling weights by falling water. A typical water machine that was supposed to move by itself forever is shown in the next drawing. The main parts of this machine are the water wheel, the spiral pipe that is inside the sloping cylinder, and a tank full of water. The machine was started by using the crank to turn the wheel. The teeth on the turning wheel meshed with a gear on the sloping cylinder and made the cylinder turn. As the cylinder turned, the spiral pipe that was in it scooped water out of the tank, and then, in turn after turn, gradually raised the water to the top. Here it flowed into a reservoir which fed a pipe that poured water over the wheel. From this point on the machine was supposed to work by itself, with the force for turning the wheel being supplied by the falling water.



The designs were many, and the systems were different, but all the perpetual motion machines had one thing in common. *None of them would work.* Growing experience with machines, and the development of the science of physics uncovered the reason why. The working of every machine is governed by an important rule. This rule is a law of nature known as the law of *conservation of energy*. The law says that energy cannot be created or destroyed. This means that the energy in a system cannot change in *amount*. It can only change in *form*. Energy exists in many different forms or disguises. Sometimes we can see it as *light*. Sometimes we can feel it as *heat*. At times it appears as *motion*. But it may also be stored or hidden in various ways. It may be the *chemical energy* hidden in coal, or the *electrical energy* stored in a condenser. It may also be lurking in the *position* of a body. For example, when we lift a ball, we use up energy in the form of the motion that lifts it. This energy is stored in the raised position of the ball. When the ball falls, it releases the energy again as motion. But the energy it releases is exactly the same as the energy that was stored in it in the first place. The energy released is just enough to raise the ball again to the same height, *provided that none of it is drawn off for some other purpose*. But here we find the reason why the machines that used falling weights could not go on forever. The moving parts of a machine rub against each other, and the rubbing or *friction* changes some of the energy of motion into heat. This change of motion into heat is a leak in the machine's energy supply. Because of this leak, some of the energy released by a falling ball is lost. But then the energy that is left is not enough to raise the ball to the same height again. If the machine is made to do any work for us, more of the machine's energy is drawn off to do this work. So unless more energy is fed into the machine, its energy supply is steadily lost. The machine must slow down, and eventually stop. The construction of a perpetual motion machine is made impossible by the law of the conservation of energy.

Several hundred years ago it made sense for people to try to trisect an angle or make a perpetual motion machine. At that time they were merely trying to do things that no one had been able to do yet. And the fact that no one had succeeded in doing them before did not mean that they would never be done. But now, after it has been proved that these things are impossible to do, it is a foolish waste of effort to try. But that doesn't seem to stop people from trying, or even believing that they have succeeded. Sometimes, as in the case of Father Callahan, they even get the public to believe that they have done something great. But the truth reaches the public sooner or later, and the only ones who remain fooled are those who insist on trying to do what has been proved impossible.